

# Differentiations of operator algebras over non-archimedean fields.

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## Abstract

Differentiations of operator algebras over non-archimedean spherically complete fields are investigated. Theorems about a differentiation being internal are demonstrated.<sup>1</sup>

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## 1 Introduction.

Differentiations of operator algebras over the complex field were investigated in [4, 13, 8]. It was shown that derivations of  $C^*$ -algebras and von Neumann algebras are internal. But the case of operator algebras over non-archimedean fields was not studied yet.

This article continues previous investigations of operator algebras over non-archimedean fields (see [1] and references therein), where their spectral theory was described. The present paper is devoted to investigations of derivations of operator algebras over infinite spherically complete fields with non-trivial non-archimedean multiplicative norms having values in  $\Gamma \cup \{0\}$ ,

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where  $\Gamma$  is a discrete multiplicative group, particularly over locally compact fields. Theorems about a differentiation being internal are demonstrated.

All results of this paper are obtained for the first time.

## 2 Differentiations of operator algebras

**1. Definitions.** Suppose that  $\mathbf{F}$  is an infinite field supplied with a non-archimedean non-trivial multiplicative norm relative to which it is complete as a uniform space. Let  $X$  be a Banach space over  $\mathbf{F}$ , denote by  $L(X) = L(X, X)$  a Banach space of all continuous  $\mathbf{F}$ -linear operators from  $X$  into  $X$ .

An algebra  $\Psi$  contained in  $L(X)$  over  $\mathbf{F}$  such that  $A^t \in \Psi$  for each  $A \in \Psi$  will be called an algebra with transposition, where a mapping  $A \mapsto A^t$  on  $\Psi$  is called a transposition, if it is  $\mathbf{F}$ -linear and  $(A^t)^t = A$  and  $(AB)^t = B^t A^t$  for each  $A, B \in \Psi$ .

A Banach algebra with transposition is called a  $T$ -algebra.

A subalgebra of  $L(X)$  will be called an operator algebra.

An operator  $A \in \Psi$  is called symmetric if  $A^t = A$ .

An  $\mathbf{F}$ -linear continuous mapping  $D : \Psi \rightarrow \Psi$  on an algebra  $\Psi$  over  $\mathbf{F}$  is called a derivation of  $\Psi$  if  $D(AB) = D(A)B + AD(B)$  for each  $A, B \in \Psi$ .

For subalgebras  $\Phi$  and  $\Psi$  of  $L(X)$  satisfying the condition if  $A \in \Phi \cup \Psi$  then  $A^t \in L(X)$  let  $T\{\Phi, \Psi\}$  denote the minimal  $T$  subalgebra in  $L(X)$  containing  $\Phi$  and  $\Psi$ .

A norm of an operator  $A \in L(X)$  is defined as

$$\|A\| := \sup_{x \neq 0} \|Ax\| / \|x\|.$$

A homomorphism  $\phi : \Psi \rightarrow L(X)$  such that it is  $\mathbf{F}$ -linear,  $\phi(aA + bB) = a\phi(A) + b\phi(B)$ , and multiplicative  $\phi(AB) = \phi(A)\phi(B)$  and  $\phi(A^t) = [\phi(A)]^t$  for each  $A, B \in \Psi$  and  $a, b \in \mathbf{F}$  is called a representation of a  $T$  algebra on  $X$ . If additionally  $\phi$  is bijective, then such representation  $\phi$  is called faithful.

**2. Remark.** Henceforth, operator  $T$ -algebras are considered. Denote by  $c_0(\alpha, \mathbf{F})$  the Banach space of all mappings  $x : \alpha \rightarrow \mathbf{F}$  satisfying the condition that for each  $\epsilon > 0$  the set  $\{j : j \in \alpha; |x_j| > \epsilon\}$  is finite, where  $c_0(\alpha, \mathbf{F})$  is supplied with norm  $\|x\| := \sup_{j \in \alpha} |x_j|$ ,  $x_j = x(j)$ ,  $\alpha$  is a set. That is either  $\Gamma_{\mathbf{F}} := \{|x| : x \in \mathbf{F} \setminus \{0\}\}$  is discrete or  $\text{card}(\alpha) < \aleph_0$ .

If  $X = c_0(\alpha, \mathbf{F})$  and  $A \in L(X)$ , then a transposed operator  $A^t$  can be defined by the equality  $A_{j,k}^t = A_{k,j}$  for each  $j, k \in \alpha$ , where  $Ae_k = \sum_{j \in \alpha} A_{k,j}e_j$  with  $A_{k,j} \in \mathbf{F}$ ,  $e_j \in c_0(\alpha, \mathbf{F})$  denotes the basic vector  $e_j(k) = \delta_{k,j}$  for each  $k \in \alpha$ ,  $\delta_{k,j} = 0$  for  $j \neq k$ , while  $\delta_{j,j} = 1$ . For  $X = c_0(\alpha, \mathbf{F})$  this operation  $A \mapsto A^t$  will serve as the transposition if  $A$  and  $A^t$  are in  $L(X)$ .

If  $\mathbf{F}$  is a spherically complete non-archimedean field with discrete multiplicative group  $\Gamma_{\mathbf{F}}$  or  $X$  is finite dimensional over  $\mathbf{F}$ , then a Banach space  $X$  over  $\mathbf{F}$  is isomorphic with  $c_0(\alpha, \mathbf{F})$  for some set  $\alpha$  (see Theorems 5.13 and 5.16 [12]). Henceforward, it is supposed that either a spherically complete field  $\mathbf{F}$  is locally compact or it contains a family  $\{\mathbf{G}_\alpha : \alpha \in \mu\}$  of locally compact subfields  $\mathbf{G}_\alpha$  such that their union is dense in  $\mathbf{F}$ , that is  $\overline{\bigcup_\alpha \mathbf{G}_\alpha} = \mathbf{F}$ , where  $\bar{A}$  denotes the completion of a subset  $A$  relative to the uniformity inherited from  $\mathbf{F}$ .

Henceforth, it is supposed that a Banach space  $X$  is isomorphic with  $c_0(\alpha, \mathbf{F})$ .

Let  $C_\infty(\Lambda, \mathbf{F})$  denote a Banach algebra of all continuous functions  $f : \Lambda \rightarrow \mathbf{F}$  such that for each  $\epsilon > 0$  there exists a compact subset  $V$  in  $\Lambda$  for which  $|f(x)| \leq \epsilon$  for every  $x \in \Lambda \setminus V$ , where  $\Lambda$  is a zero-dimensional locally compact Hausdorff space, while  $C(\Lambda, \mathbf{F})$  denotes the algebra of all continuous functions  $f : \Lambda \rightarrow \mathbf{F}$ . If a Banach algebra  $\Psi$  is isomorphic with  $C_\infty(\Lambda, \mathbf{F})$ , then it is called a  $C$ -algebra.

If  $\mathbf{F}$  is a field with a multiplicative norm and  $\Lambda$  is a subset in  $\mathbf{F}$ , a space of all continuous functions  $f : \Lambda \rightarrow \mathbf{F}$  so that for each  $\epsilon > 0$  a positive number  $0 < r < \infty$  exists for which  $|f(x)| < \epsilon$  for each  $x \in \Lambda$  with  $|x| > r$  is denoted by  $C_\infty(\Lambda, \mathbf{F})$ . That is  $C_\infty(\Lambda, \mathbf{F})$  is a space of continuous functions tending to zero at infinity.

Evidently, each  $C$ -algebra is a  $T$ -algebra.

If a Banach space  $X$  is over a spherically complete field  $\mathbf{F}$  and  $X^*$  is its topological dual Banach space, i.e. of all continuous  $\mathbf{F}$ -linear functionals  $y^* : X \rightarrow \mathbf{F}$ , then each  $A \in L(X, Y)$  has an adjoint operator  $A^* : Y^* \rightarrow X^*$ , where  $Y$  is a Banach space over  $\mathbf{F}$ ,  $A^* \in L(Y^*, X^*)$ . On the other hand,  $X^*$  is the Banach space over the spherically complete field  $\mathbf{F}$  and hence isomorphic with  $c_0(\beta, \mathbf{F})$  for some set  $\beta$ . But each vector  $x \in X = c_0(\alpha, \mathbf{F})$  gives rise to

a continuous  $\mathbf{F}$ -linear functional  $x^*z := \sum_{j \in \alpha} x_j z_j$  for each  $z \in X$ . Therefore, the natural embedding  $X \hookrightarrow X^*$  exists, that is  $\alpha \subset \beta$ . This implies, that the operation  $L(X) \ni A \mapsto A^* \in L(X^*)$  can be considered as an extension of  $A \mapsto A^t$  from  $X$  onto  $X^*$  for each  $A \in L(X, X) = L(X)$  (see also Chapter 3 in [12]).

For a Banach space  $X$  over a spherically complete field  $\mathbf{F}$  each closed linear subspace  $Y$  is orthocomplemented in accordance with Theorems 5.13 and 5.16 [12]. Therefore, in such case it is written below for short a projection  $\pi_Y : X \rightarrow Y$  instead of an orthoprojection, where  $\pi_Y(X) = Y$  (see also [1]).

**3. Lemma.** *If  $\Phi$  is a  $C$ -algebra over  $\mathbf{F}$  and  $D$  is its differentiation, then  $D = 0$  on it.*

**Proof.** In the space  $C_\infty(\Lambda, \mathbf{F})$  an  $\mathbf{F}$ -linear subspace of simple functions

$$f(x) = \sum_{j=1}^n a_j \chi_{B_j}$$

is dense, where  $a_j \in \mathbf{F}$ ,  $B_j$  is a clopen subset in  $\Lambda$ ,  $\chi_B$  denotes the characteristic function of a subset  $B$  in  $\Lambda$ , that is  $\chi_B(x) = 1$  for each  $x \in B$ , while  $\chi_B(x) = 0$  for any  $x \in \Lambda \setminus B$ . Then  $D(\chi_B) = D(\chi_B^2) = 2D(\chi_B)\chi_B$ , consequently,  $D(\chi_B)(1 - 2\chi_B) = 0$  and hence  $D(\chi_B) = 0$ . A differentiation  $D$  is  $\mathbf{F}$ -linear and continuous, consequently,  $D(f) = 0$  for each  $f \in C_\infty(\Lambda, \mathbf{F})$ .

**4. Lemma.** *If  $A \in L(X)$ , where  $X$  is Banach space over a locally compact field  $\mathbf{F}$ , and an operator  $A$  is such that  $\overline{\mathbf{F}(A)}$  is a least closed  $C$ -subalgebra of  $L(X)$  containing  $A$ , and a spectrum of  $\overline{\mathbf{F}(A)}$  is contained in a closed ball  $B(\mathbf{F}, 0, \|A\|)$  containing 0 in  $\mathbf{F}$  of radius  $\|A\|$ ,  $D : \overline{\mathbf{F}(A)} \rightarrow L(X)$  and  $\pi_{\overline{\mathbf{F}(A)}} D : \overline{\mathbf{F}(A)} \rightarrow \overline{\mathbf{F}(A)}$  are differentiations, then  $\pi_{\overline{\mathbf{F}(A)}} D B = 0$  for each  $B \in \overline{\mathbf{F}(A)}$ , where  $\pi_\Psi : L(X) \rightarrow \Psi$  denotes an  $\mathbf{F}$ -linear projection on a closed subalgebra  $\Psi$  in  $L(X)$ .*

**Proof.** The field  $\mathbf{F}$  is locally compact, consequently, it is spherically complete. Therefore, the Banach subspace  $\overline{\mathbf{F}(A)}$  in  $L(X)$  is orthocomplemented in the non-archimedean sense and the continuous  $\mathbf{F}$ -linear projection  $\pi_{\overline{\mathbf{F}(A)}}$  exists (see Chapter 5 [12]). Take a closed ball  $B(\mathbf{F}, 0, \|A\|)$  in the field  $\mathbf{F}$ , where  $B(Y, z, r) := \{x \in Y : \rho(x, z) \leq r\}$  denotes a closed ball with center  $z$  of radius  $0 < r$  in a metric space  $Y$  with a metric  $\rho$ . Since the field  $\mathbf{F}$  is locally compact, this ball  $B(\mathbf{F}, 0, \|A\|)$  is compact. So the  $C$ -

algebra  $C(B(\mathbf{F}, 0, \|A\|), \mathbf{F})$  of all continuous functions  $f : B(\mathbf{F}, 0, \|A\|) \rightarrow \mathbf{F}$  exists. For each polynomial  $P_n(x)$  of degree  $n$  on  $B(\mathbf{F}, 0, \|A\|)$  the corresponding operator  $P_n(A)$  is defined, where  $A^0 = I$  is the unit operator,  $A^n x = A(A^{n-1}x)$  for each  $x \in X$ . The  $\mathbf{F}$ -linear space of polynomials is dense in  $C(B(\mathbf{F}, 0, \|A\|), \mathbf{F})$  in accordance with Kaplansky's theorem 43.3 [14]. By the conditions of this lemma a spectrum of a  $C$ -algebra  $\overline{\mathbf{F}(A)}$  is contained in a closed ball  $B(\mathbf{F}, 0, \|A\|)$ . Therefore,  $f(A)$  is defined for each continuous function  $f : B(\mathbf{F}, 0, \|A\|) \rightarrow \mathbf{F}$  and  $\overline{\mathbf{F}(A)}$  is contained in  $C(B(\mathbf{F}, 0, \|A\|), \mathbf{F})$  as the closed subalgebra. Certainly,  $\pi_{\overline{\mathbf{F}(A)}} DA \in \overline{\mathbf{F}(A)}$  and  $\pi_{\overline{\mathbf{F}(A)}} D(AB) = \pi_{\overline{\mathbf{F}(A)}} (DA)B + A\pi_{\overline{\mathbf{F}(A)}} DB$  for each  $A, B \in \overline{\mathbf{F}(A)}$ , hence  $\pi_{\overline{\mathbf{F}(A)}} D$  is the continuous differentiation on  $\overline{\mathbf{F}(A)}$ , since the operators  $\pi_{\overline{\mathbf{F}(A)}}$  and  $D$  are continuous. A closed subalgebra of a  $C$ -algebra is a  $C$ -algebra by Corollary 6.13 [12]. Therefore, by the preceding lemma the differentiation  $\pi_{\overline{\mathbf{F}(A)}}$  on  $\overline{\mathbf{F}(A)}$  is degenerate.

**5. Definition.** Let  $\rho : \Psi \rightarrow \mathbf{F}$  be a linear continuous functional on a  $T$ -algebra  $\Psi$  over  $\mathbf{F}$ . If  $\rho(A^t) = \rho(A)$  for each  $A \in \Psi$ , then  $\rho$  will be called symmetric. If a symmetric continuous functional  $\rho$  is such that  $\rho(I) = 1$ , then  $\rho$  is called a state of  $\Psi$ . A state  $\rho$  of a  $T$ -algebra  $\Psi$  is definite on a symmetric operator  $A$ , when  $\rho(A^n) = [\rho(A)]^n$  for every natural number  $n$ .

A functional  $\rho$  is called multiplicative on a  $T$ -algebra  $\Psi$ , if  $\rho(AB) = \rho(A)\rho(B)$  for each  $A, B \in \Psi$ .

**6. Lemma.** Let  $A$  be an operator in a Banach algebra  $\Psi$  over a locally compact field  $\mathbf{F}$  and let a continuous linear functional  $\rho : \overline{\mathbf{F}(A)} \rightarrow \mathbf{F}$  be multiplicative on  $A^n$  for each  $n \in \mathbf{N}$ . Suppose that  $\rho$  has a  $\mathbf{K}$ -linear extension on  $C(B(\mathbf{F}, 0, \|A\|), \mathbf{K})$ , where a field  $\mathbf{K}$  is an extension of a field  $\mathbf{F}$ . Then  $\rho(f(A)) = f(\rho(A))$  for each  $f \in C(B(\mathbf{F}, 0, \|A\|), \mathbf{K})$ .

**Proof.** From  $\rho(A^n) = [\rho(A)]^n$  for every  $n \in \mathbf{N}$  it follows that  $\rho$  is multiplicative on the Banach algebra  $\overline{\mathbf{F}(A)}$  generated by  $A$ . But  $\overline{\mathbf{F}(A)}$  is the Banach algebra having an isometric embedding into  $C(B(\mathbf{F}, 0, \|A\|), \mathbf{F})$ . For each polynomial  $P_m(x)$  on  $B(\mathbf{F}, 0, \|A\|)$  with values in  $\mathbf{K}$  due to multiplicativity and  $\mathbf{F}$ -linearity of  $\rho$  one gets  $\rho(P_m(A)) = P_m(\rho(A))$ . From Kaplansky's theorem and continuity of  $\rho$  it follows that  $\rho(f(A)) = f(\rho(A))$  for each continuous function  $f : B(\mathbf{F}, 0, \|A\|) \rightarrow \mathbf{K}$ .

**7. Lemma.** *Let  $A$  be a symmetric operator in a  $T$ -algebra  $\Psi$  over a locally compact field  $\mathbf{F}$  and let  $\mathbf{K}$  be its extension so that  $\sqrt[n]{x} \in \mathbf{K}$  for every  $x \in \mathbf{F}$  and for each natural number  $m \geq 2$ , then for a marked natural number  $n \geq 2$  there exists  $B \in \Psi_{\mathbf{K}}$  such that  $B^n = A$ , where  $\Psi_{\mathbf{K}}$  is an extension of an algebra  $\Psi$  over  $\mathbf{K}$ .*

**Proof.** Suppose that a set  $\alpha$  is infinite. A net of projection operators  $\pi_\gamma$  on finite dimensional subspaces  $c_0(\gamma, \mathbf{F})$  in  $c_0(\alpha, \mathbf{F})$  exists, where  $\gamma$  are finite subsets in  $\alpha$  and their family  $\Upsilon$  is partially ordered by inclusion. Then  $\mathcal{B} = \{\beta : \beta = \alpha \setminus \gamma; \gamma \in \Upsilon\}$  is a filter base. For each  $x \in X = c_0(\alpha, \mathbf{F})$  the limit  $\lim_{\mathcal{B}} \pi_\gamma x = x$  exists. Therefore,  $\lim_{\mathcal{B}} \pi_\gamma A \pi_\gamma x = Ax$  for each  $x \in X$ . Each operator  $\pi_\gamma A \pi_\gamma$  is compact from  $X$  into  $X$ . In view the decomposition theorem of compact operators (see Lemma 1 and Note 2 of Appendix A in [9]) it has the decomposition

$$(1) \pi_\gamma A \pi_\gamma = C_\gamma^{-1} \Lambda_\gamma C_\gamma \text{ over } \mathbf{K},$$

where  $C_\gamma$  is an invertible operator on  $c_0(\gamma, \mathbf{K})$  and  $\Lambda_\gamma$  is a diagonal operator on  $c_0(\alpha, \mathbf{K})$  relative to its standard base, moreover,  $C_\gamma^t = C_\gamma^{-1}$  for a symmetric operator  $\pi_\gamma A \pi_\gamma$ . The latter decomposition automatically encompasses the case of finite  $\alpha$  also. Thus  $P_n(\pi_\gamma A \pi_\gamma)$  is correctly defined for each polynomial  $P_n$  on  $\mathbf{F}$  with values in  $\mathbf{K}$  and  $\lim_{\mathcal{B}} P_n(\pi_\gamma A \pi_\gamma)x = P_n(A)x$  for every vector  $x \in X$ .

From the embedding  $\overline{\mathbf{F}(A)} \hookrightarrow C(B(\mathbf{F}, 0, \|A\|), \mathbf{F})$  (see §6) and the continuity of the function  $\mathbf{F} \ni x \mapsto \sqrt[n]{x} \in \mathbf{K}$  it follows that  $B = \sqrt[n]{A} \in \overline{\mathbf{K}(A)}$ . The latter algebra is contained in  $\Psi_{\mathbf{K}}$ .

**8. Lemma.** *Let  $D$  be a derivation operator on a  $T$ -algebra  $\Psi$  over a field  $\mathbf{F}$ , let also  $\mathbf{K}$  be an extension of  $\mathbf{F}$  complete relative to its uniformity. Then  $D$  has a continuous  $\mathbf{K}$ -linear extension on  $\Psi_{\mathbf{K}}$  as a derivation operator.*

**Proof.** A field  $\mathbf{K}$  contains  $\mathbf{F}$  as a subfield, a multiplicative non-archimedean norm on  $\mathbf{F}$  has a multiplicative non-archimedean extension on  $\mathbf{K}$  (see [12, 14]). The completion  $\tilde{\mathbf{K}}$  of  $\mathbf{K}$  relative to this norm is a field complete relative to the norm. A Banach space  $X$  over a field  $\mathbf{F}$  has an  $\mathbf{F}$ -linear continuous embedding into  $X_{\tilde{\mathbf{K}}}$ , where  $X = c_0(\alpha, \mathbf{F})$  and  $X_{\tilde{\mathbf{K}}} = c_0(\alpha, \tilde{\mathbf{K}})$ .

Then the closure of the  $\tilde{\mathbf{K}}$ -linear span of  $\Psi$  in  $L(X_{\tilde{\mathbf{K}}})$  gives  $\Psi_{\tilde{\mathbf{K}}}$  such that the embedding  $\Psi \hookrightarrow \Psi_{\tilde{\mathbf{K}}}$  is continuous relative to the operator norm. By

the condition of this lemma  $\mathbf{K}$  is complete relative to its uniformity, hence  $\tilde{\mathbf{K}} = \mathbf{K}$  and  $\Psi_{\mathbf{K}} = \Psi_{\tilde{\mathbf{K}}}$ .

Put  $DbA = bDA$  for each  $b \in \tilde{\mathbf{K}}$  and  $A \in \Psi$ . Therefore,  $D(bAB) = bD(AB) = b(DA)B + bADB$  for any  $b \in \tilde{\mathbf{K}}$  and  $A, B \in \Psi$ . Moreover,  $\|D(bAB)\| \leq |b| \max(\|(DA)B\|, \|A(DB)\|)$  and  $(DbA + tB) \leq \max(|b|\|DA\|, |t|\|DB\|)$  for each  $b, t \in \tilde{\mathbf{K}}$  and  $A, B \in \Psi$ , consequently,  $D$  has a continuous  $\tilde{\mathbf{K}}$ -linear extension as a derivation operator on the  $\tilde{\mathbf{K}}$ -linear span of  $\Psi$  in  $L(X_{\tilde{\mathbf{K}}})$  and hence on its completion  $\Psi_{\tilde{\mathbf{K}}}$ .

**9. Lemma.** *Suppose that  $D$  is a derivation of a  $T$ -algebra  $\Psi$  over a field  $\mathbf{F}$  having an extension up to a derivation on  $\Psi_{\mathbf{K}}$ , where a field  $\mathbf{K}$  is an extension of  $\mathbf{F}$ . Let  $\rho$  be a definite state on a symmetric operator  $A$  and  $A = B^2$  for some symmetric operator  $B \in \overline{\mathbf{K}(A)}$  and  $\rho$  has an extension as a state on  $\Psi_{\mathbf{K}}$  and either  $DB \in \overline{\mathbf{K}(A)}$  or  $\rho(B(DB)) = \rho(B)\rho(DB)$  and  $\rho((DB)B) = \rho(DB)\rho(B)$ . Then  $\rho(DA) = 0$ .*

**Proof.** The differentiation operator  $D$  is  $\mathbf{F}$ -linear, hence  $DA = D(A - \rho(A)I)$ , since  $DI = 0$ . Therefore, without loss of generality it is sufficient to consider the case  $\rho(A) = 0$ , since  $\rho(A - \rho(A)I) = 0$ . Consider an operator  $B = A^{1/2}$ , i.e  $A = B^2$ . A state  $\rho$  is multiplicative on the  $T$ -algebra  $\overline{\mathbf{F}(A)}$  generated by  $A$ , since  $\rho(A^n) = [\rho(A)]^n$  for each natural number  $n$ . Thus  $\rho(B) = 0$  and hence  $\rho(DA) = \rho((DB)B) + \rho(B(DB)) = \rho(B)\rho(DB) + \rho(DB)\rho(B) = 0$ , since either  $DB \in \overline{\mathbf{K}(A)}$  or  $\rho(B(DB)) = \rho(B)\rho(DB)$  and  $\rho((DB)B) = \rho(DB)\rho(B)$ .

**10. Theorem.** *Suppose that  $D$  is a derivation of a  $T$ -algebra  $\Psi$  over a locally compact field  $\mathbf{F}$  such that  $\sqrt{x} \in \mathbf{K}$  for each  $x \in \mathbf{F}$ , where a field  $\mathbf{K}$  is an extension of  $\mathbf{F}$ , and that  $\mathcal{Z}$  is a center of  $\Psi$ . Then  $D$  annihilates  $\mathcal{Z}$ .*

**Proof.** Since  $\Psi$  is a Banach algebra, its center  $\mathcal{Z}$  is closed in  $\Psi$ . The field  $\mathbf{F}$  is spherically complete, consequently,  $\mathcal{Z}$  is complemented in  $\Psi$ . Take  $A \in \mathcal{Z}$  and consider  $\overline{\mathbf{F}(A)}$  which is complemented in  $\mathcal{Z}$  and hence in  $\Psi$ . Any element  $A \in \Psi$  can be presented as  $A = A_1 - A_2$ , where  $A_1^t = A_1$  is symmetric and  $A_2^t = -A_2$  is antisymmetric,  $A_1 = \frac{A^t + A}{2}$ ,  $A_2 = \frac{A^t - A}{2}$ . If  $A \in \mathcal{Z}$ , then  $A_1, A_2 \in \mathcal{Z}$ . Then  $\sqrt{-1} = i \in \mathbf{K}$  and  $(iA_2)^t = -A_2$ , where  $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  over the field  $\mathbf{F}$ , when  $i \notin \mathbf{F}$ . On the other hand, if  $i \in \mathbf{F}$ , one can consider  $\Psi \oplus \Psi$  on  $X \oplus X$ , where  $\begin{pmatrix} 0 & A_2 \\ -A_2 & 0 \end{pmatrix}$  is symmetric, when  $A_2$  is antisymmetric.

Therefore, it is sufficient to consider the case of symmetric  $A \in \mathcal{Z}$ .

Choose any multiplicative  $\mathbf{F}$ -linear continuous functional  $\rho$  on  $\overline{\mathbf{F}(A)}$  so that  $\rho(I) = 1$  and  $\rho(A) \neq 0$ . Consider a projection  $\pi_Y DA$  of  $DA$  onto a Banach subspace  $Y = \Psi \ominus \overline{\mathbf{F}(A)}$ , i.e.  $\Psi = Y \oplus \overline{\mathbf{F}(A)}$ . Take any continuous extension of  $\rho$  so that  $\rho(\pi_Y DA) \neq 0$  and such that  $\rho$  is multiplicative on  $\overline{\mathbf{F}(A, \pi_Y DA)}$ , where  $\overline{\mathbf{F}(A_1, \dots, A_n)}$  denotes a minimal closed subalgebra of  $\Psi$  containing elements  $A_1, \dots, A_n \in \Psi$ . This is possible, since  $\overline{\mathbf{F}(A, \pi_Y DA)}$  is the algebra with two commuting generators  $[A, \pi_Y DA] = 0$ . Moreover, the inclusion  $A \in \mathcal{Z}$  implies  $\overline{\mathbf{F}(A)} \subset \mathcal{Z}$  and  $\Psi/\mathcal{Z} = (\Psi/\overline{\mathbf{F}(A)})/(\mathcal{Z}/\overline{\mathbf{F}(A)})$  and  $(\pi_Y DA) + \overline{\mathbf{F}(A)} = \theta(\pi_Y DA) = DA + \overline{\mathbf{F}(A)}$ , where  $\theta : \Psi \rightarrow \Psi/\overline{\mathbf{F}(A)}$  denotes the quotient mapping. Then we also get  $\theta(C) = C + \overline{\mathbf{F}(A)}$  and  $D(AB) + \overline{\mathbf{F}(A)} = (DA)B + A(DB) + \overline{\mathbf{F}(A)} = (DA + \overline{\mathbf{F}(A)})(B + \overline{\mathbf{F}(A)}) + (A + \overline{\mathbf{F}(A)})(DB + \overline{\mathbf{F}(A)}) = \theta(D(AB)) = \theta(DA)\theta(B) + \theta(A)\theta(DB)$ , consequently,  $\theta \circ D$  is the differentiation on the quotient algebra  $\Psi/\overline{\mathbf{F}(A)}$ .

If  $V \in \overline{\mathbf{F}(\pi_Y DA)} \ominus \overline{\mathbf{F}(A)}$  is a non zero element and  $V^n \in \overline{\mathbf{F}(A)}$  for some natural number  $n \geq 2$ , then take an algebraically closed field  $\mathbf{K}$  containing  $\mathbf{F}$  so that  $\sqrt[n]{x} \in \mathbf{K}$  for each  $x \in \mathbf{F}$ . Therefore,  $V \in \overline{\mathbf{K}(A)}$  and one can take  $\rho(V) = \sqrt[n]{f(\rho(A))}$  with  $Q = V^n = f(A) \in \overline{\mathbf{F}(A)}$ , where  $f$  is a continuous function from  $B(\mathbf{F}, 0, 1)$  into  $\mathbf{F}$  (see Lemmas 6 and 7). Thus it remains to treat the variant when  $V^n \notin \overline{\mathbf{F}(A)}$  for each natural number  $n$ .

For this it is sufficient to choose a multiplicative extension of  $\rho$  on  $\overline{\mathbf{F}(\pi_Y DA)} \ominus \overline{\mathbf{F}(A)}$  putting  $\rho(V^n) = [\rho(V)]^n \neq 0$  for each natural number  $n$  and for some non zero element  $V \in \overline{\mathbf{F}(\pi_Y DA)} \ominus \overline{\mathbf{F}(A)}$ . Indeed, without loss of generality using multiplication on scalars  $A \mapsto bA$  for  $b \in \mathbf{F} \setminus \{0\}$  it is possible to restrict on the case  $\max(\|A\|, \|V\|) < 1$  and choose  $0 < |\rho(A)| \leq \|A\|$  and  $0 < |\rho(V)| \leq \|V\|$ . The Banach subspace  $\overline{\mathbf{F}(A, \pi_Y DA)}$  is closed in  $\Psi$ , consequently, by the non-archimedean Hahn-Banach theorem over the spherically complete field  $\mathbf{F}$  a functional  $\rho$  has a continuous extension on  $\Psi$  (see [12] and §8.203 in [10]).

The family of such functionals  $\rho$  separates different elements of  $\Psi$  and  $Q \in \Psi \ominus \overline{\mathbf{F}(A, \pi_Y DA)}$ , hence  $\rho(DA) = 0$  for each such  $\rho$  if and only if  $DA = 0$ .

Applying Lemmas 6-9 we get the statement of this theorem.



**11. Definition.** If  $\Psi$  is a  $T$ -algebra on a Banach space  $X$  over a field  $\mathbf{F}$ , its strong topology is characterized by a base of neighborhoods  $V_{x_1, \dots, x_n; \epsilon} := \{A \in \Psi : \|Ax_j\| < \epsilon \ \forall j = 1, \dots, n\}$  of zero, where  $x_1, \dots, x_n \in X$ ,  $\epsilon > 0$ ,  $n \in \mathbf{N}$ . If a field  $\mathbf{F}$  is spherically complete and  $X^*$  is a topological dual space of  $X$ , a weak topology on  $\Psi$  is given by a base of neighborhoods  $W_{x_1, \dots, x_n; y_1, \dots, y_n; \epsilon} := \{A \in \Psi : |y_j Ax_j| < \epsilon \ \forall j = 1, \dots, n\}$  of zero, where  $x_1, \dots, x_n \in X$ ,  $y_1, \dots, y_n \in X^*$ ,  $\epsilon > 0$ ,  $n \in \mathbf{N}$ . Denote by  $\bar{\Psi}$  the completion of  $\Psi$  relative to the weak topology.

**12. Lemma.** Suppose that  $D$  is a derivation of a  $T$ -algebra  $\Psi$  on a Banach space  $X$  over a spherically complete field  $\mathbf{F}$ . Then a unique weakly continuous extension  $\bar{D}$  of  $D$  on  $\bar{\Psi}$  exists.

**Proof.** The mappings  $A \mapsto \frac{A^t + A}{2} =: A_1$  and  $A \mapsto \frac{A^t - A}{2} =: A_2$  are continuous on a  $T$ -algebra  $\Psi$ . An extension  $\mathbf{K}$  from Lemma 7 of a spherically complete field  $\mathbf{F}$  can be considered as an  $\mathbf{F}$ -linear space. By Lemma 8  $D$  has a continuous extension on  $\Psi_{\mathbf{K}}$  as a derivation operator. As in Lemma 10 it is sufficient to consider a symmetric operator  $A$ .

Put  $\mathbf{S} := \{A \in \Psi : \|A\| \leq 1, A^t = A\}$  to be the unit ball of symmetric operators. Then the mapping  $\mathbf{S} \ni A \mapsto y(D(A^2)x) = y(ADAx + (DA)Ax)$  is strongly continuous at zero, since  $|y(ADAx + (DA)Ax)| \leq \|D\| \max(\|Ax\| \|y\|; \|x\| \|Ay\|)$ , where  $x, y \in X$  and  $X$  is embedded into  $X^*$ . On the other side, the mapping  $\mathbf{S} \ni A \mapsto A^{1/2}$  is strongly continuous at zero, since  $\|A^{1/2}x\| \leq \|Ax\| \|x\|$  due to Formula 7(1), where  $x \in X$ . Thus  $A \mapsto y(DA_1x) - y(DA_2x) = y(DAx)$  is strongly continuous at zero on  $\mathbf{S}$ . This implies that  $H := \mathbf{S} \cap q^{-1}(B(\mathbf{F}, 0, r))$  is strongly closed in  $\mathbf{S}$ , where  $q(A) := y(DAx)$  for some marked vectors  $x, y \in X$ ,  $0 < r < \infty$ .

Recall that a subset  $U$  of a topological  $\mathbf{F}$ -linear space  $Q$  is called absolutely  $\mathbf{F}$ -convex if  $B(\mathbf{F}, 0, 1)U + B(\mathbf{F}, 0, 1)U \subset U$ .

The norm on  $\mathbf{F}$  is non-archimedean, i.e.  $|a + b| \leq \max(|a|, |b|)$  for each  $a, b \in \mathbf{F}$ . It can be lightly seen, that the set  $H$  is absolutely  $\mathbf{F}$ -convex and strongly closed, consequently,  $H$  is weakly closed in  $\mathbf{S}$ . Indeed, if a net  $T_n \in H$  strongly converges to  $T \in H$ , then  $T_n - T \in H$  for each  $n$  and hence the net  $(T_n - T)$  strongly converges to zero. Therefore,  $y((T_n - T)x)$  converges to zero for each  $x \in X$  and  $y \in X^*$ .

By the non-archimedean Hahn-Banach theorem 8.203 [10] the set  $H$  is closed relative to a weak topology with functionals from  $X$ , since the set of continuous  $\mathbf{F}$ -linear functionals  $y \in X^*$  separates points in  $X$ , i.e. from  $\lim_n y((T_n - T)x) = 0$  for each  $y \in X^*$  it follows  $\lim_n y((T_n - T)x) = 0$  for every  $y \in X^*$ , where  $T, T_n \in H$ .

Therefore, the derivation  $D$  is weakly continuous on  $B(\Phi, 0, 1)$ , since the mapping  $B(\Phi, 0, 1) \ni A \mapsto y(DAx)$  is continuous for each marked  $x, y \in X$  and the derivation operator  $D$  is  $\mathbf{F}$ -linear. This means that  $D$  is uniformly continuous relative to the weak uniformity on  $B(\Phi, 0, 1)$  and implies that  $D$  has a continuous extension on  $\overline{B(\Phi, 0, 1)}$  and hence on  $\bar{\Phi}$  with range in  $\bar{\Phi}$  by Theorem 8.3.10 [2], since  $\overline{B(\Phi, 0, 1)}$  is the closed absorbing set in  $\bar{\Phi}$ .

This extension is  $\mathbf{F}$ -linear as well, since

$\lim_n D(bT_n + H_n) = b \lim_n DT_n + \lim_n DH_n$  for each  $b \in \mathbf{F}$  and  $T_n, H_n \in \Phi$  with  $\lim_n T_n = T \in \bar{\Phi}$  and  $\lim_n H_n = H \in \bar{\Phi}$ . Moreover,  $y(D(T_n H_n)x) = y((DT_n)H_n x) + y(T_n(DH_n)x)$  for each  $x \in X$  and  $y \in X^*$ , consequently,

$\lim_n \lim_k y((DT_n)H_k x) + y(T_n(DH_k)x) - y(D(T_n H_k)x) = y((DT)Hx) + y(T(DH)x) - y(D(TH)x) = 0$ , since  $D$  is the bounded operator on  $\Phi$  and weakly continuous on  $\bar{\Phi}$ , hence  $D$  is the derivation on  $\bar{\Phi}$  as well.

**13. Definitions.** A  $T$ -algebra of bounded operators on a Banach space  $X$  over a spherically complete field  $\mathbf{F}$  closed relative to the weak operator topology and containing the unit operator will be called a  $W^t$ -algebra. For an operator  $A \in L(X)$  and a  $W^t$ -algebra  $\Psi$  let  $\overline{\text{co}}_\Psi(A)$  denote the closure relative to the weak operator topology of finite combinations  $b_1 B_1 + \dots + b_n B_n$  of operators  $B_j = V_j A V_j^t$ , where  $V_j$  is an isometry operator on  $X$  for every  $j$ , i.e.  $\|V_j x\| = \|x\|$  for each  $x \in X$ ,  $b_1, \dots, b_n \in B(\mathbf{F}, 0, 1)$ .

If  $\Upsilon$  is a family of operators in  $L(X)$ , then  $\Upsilon' := \{C : C \in L(X); [C, T] = 0 \ \forall T \in \Upsilon\}$  denotes the commutant of  $\Upsilon$ , where  $[C, T] = CT - TC$  is the commutator of two operators.

A center  $Z(\Psi)$  of an algebra  $\Psi$  is a set of all its elements commuting with each element in  $\Psi$ . An element  $A \in Z(\Psi)$  in the center is called central.

**14. Lemma.** *Let  $A$  be a linear continuous operator  $A : X \rightarrow X$  on a Banach space over a spherically complete field  $\mathbf{F}$  and let  $\mathbf{G}$  be a locally compact field contained in  $\mathbf{F}$ . Suppose that  $f \in C_\infty(\mathbf{F}, \mathbf{F})$  is a continuous*

function tending to zero at infinity the restriction of which  $f|_{\mathbf{G}}$  belongs to  $C_{\infty}(\mathbf{G}, \mathbf{G})$ . Then a linear continuous bounded operator  $f(A) \in L(X)$  exists.

**Proof.** The field  $\mathbf{Q}_p$  of  $p$ -adic numbers is locally compact. Let  $\mathbf{G}$  be a locally compact field so that  $\mathbf{Q}_p \subset \mathbf{G} \subset \mathbf{F}$ , i.e. either a locally compact field  $\mathbf{G}$  containing  $\mathbf{Q}_p$  or the  $p$ -adic field itself. Then each  $\mathbf{F}$ -linear operator is also  $\mathbf{G}$  linear.

Let  $X_{\mathbf{G}}$  denote the Banach space over  $\mathbf{G}$  obtained from the Banach space  $X$  over  $\mathbf{F}$  considering  $\mathbf{F}$  as the Banach space over  $\mathbf{G}$ , i.e. by the restriction of the field of scalars. Take  $P$  a projection  $P \in \mathbf{P}_{\mathbf{G}}$  on a finite-dimensional over  $\mathbf{G}$  subspace in  $X_{\mathbf{G}}$  with  $\mathbf{P}_{\mathbf{G}}$  denoting the family of all projections having finite dimensional ranges partially ordered by inclusion of their ranges in  $X_{\mathbf{G}}$ . Then each operator  $PAP$  can be reduced to the diagonal form

$$(1) PAP = CTE$$

over  $\mathbf{G}$  by a lower and upper triangular operators  $C$  and  $E$  respectively invertible on  $PX$  with diagonal operator  $T$  such that  $(C - I)$  and  $(E - I)$  are nilpotent operators on  $PX_{\mathbf{G}}$  (see Lemma 1 of Appendix A in [9]).

In accordance with E. Zermelo's theorem on each set  $\Lambda$  a relation exists, which well orders  $\Lambda$  (see [2]). Suppose that  $P_{\beta}$  is a family of projections on a Banach space over a spherically complete field  $\mathbf{F}$ , where  $\beta \in \Lambda$  and a set  $\Lambda$  is well ordered and  $P_{\alpha} \leq P_{\beta}$  for each  $\alpha \leq \beta$ . Denote by  $\bigwedge_{\alpha \in \Lambda} P_{\alpha}$  an projection from  $X$  onto the subspace  $\bigcap_{\alpha \in \Lambda} P_{\alpha}X$ , while defining  $\bigvee_{\alpha \in \Lambda} P_{\alpha} := I - \bigwedge_{\alpha \in \Lambda} (I - P_{\alpha})$ , where  $I$  is the unit operator on  $X$ ,  $Ix = x$  for each  $x \in X$ . Then the family  $Q_{\alpha} := P_{\alpha} - \bigvee_{\beta < \alpha} P_{\beta}$  consists of mutually orthogonal projections on  $X$  such that its sum is  $\bigvee_{\beta \in \Lambda} Q_{\beta} = \bigvee_{\beta \in \Lambda} P_{\beta} =: P$ .

Indeed,  $Q_{\beta} \perp P_{\alpha}$  are orthogonal for each  $\alpha < \beta$  and  $Q_{\beta} \perp Q_{\alpha}$ , since  $Q_{\alpha} \subseteq P_{\alpha}$ , i.e.  $Q_{\alpha}X \subseteq P_{\alpha}X$ . Therefore,  $\bigvee_{\alpha \in \Lambda} Q_{\alpha} \subseteq P$ . If  $\alpha_1$  is the least element of  $\Lambda$ , then  $P_{\alpha_1} = Q_{\alpha_1}$ . Suppose that

$$P_{\beta} = \bigvee_{\alpha \leq \beta, \alpha \in \Lambda} Q_{\alpha}$$

for each  $\beta < \gamma \in \Lambda$ . From the definition of  $Q_{\gamma}$  it follows, that

$$Q_{\gamma} = P_{\gamma} - \bigvee_{\alpha < \gamma, \alpha \in \Lambda} Q_{\alpha}, \text{ consequently,}$$

$$P_{\gamma} = I - \bigwedge_{\alpha \leq \gamma; \alpha \in \Lambda} (I - Q_{\alpha}) = \bigvee_{\alpha \leq \gamma, \alpha \in \Lambda} Q_{\alpha}.$$

Thus by transfinite induction the latter equality is fulfilled for each  $\gamma \in \Lambda$ , hence  $P \subseteq \bigvee_{\alpha \in \Lambda} Q_{\alpha}$ , together with the opposite inclusion this implies  $P =$

$$\bigvee_{\alpha \in \Lambda} Q_\alpha.$$

The field  $\mathbf{F}$  is spherically complete and considered as a Banach space over  $\mathbf{G}$  is isomorphic with  $c_0(\beta, \mathbf{G})$  for some set  $\beta$  by Theorems 5.13 and 5.16 [12]. Therefore,  $\lim_{\mathcal{P}_{\mathbf{G}}} PAPx = Ax$  for each  $x \in X$ . To an operator  $Y \in L(X)$  an operator  $Y_{\mathbf{G}} \in L(X_{\mathbf{G}})$  corresponds such that to each matrix element  $e_j^* Y e_k$  over  $\mathbf{F}$  an operator block on  $c_0(\beta, \mathbf{G})$  is posed.

Then  $C - I$  and  $E - I$  are nilpotent operators such that  $(C - I)^l = 0$  and  $(E - I)^l = 0$  for each  $l \geq m$ , where  $m$  is an order of a square  $m \times m$  matrix with entries in  $\mathbf{G}$ , i.e.  $m = \dim_{\mathbf{G}} PX_{\mathbf{G}}$  is a dimension of  $PX_{\mathbf{G}}$  over the field  $\mathbf{G}$ , operators  $C$  and  $E$  are as in Formula 14(1). Therefore,

$$(2) \quad C^k = \sum_{0 \leq h \leq \min(m, k)} \binom{k}{h} (C - I)^h,$$

where  $(C - I)^0 = I$  is the unit operator, as usually  $\binom{k}{h} = k!/(h!(k-h)!)$  denotes the binomial coefficient. Since  $\binom{k}{h}$  are integers, it follows that  $|\binom{k}{h}|_{\mathbf{G}} \leq 1$  and hence  $\|S(C)\| \leq \sup_{0 \leq h \leq \min(m, n)} |s_h| \|C - I\|^h < \infty$  for each polynomial

$$(3) \quad S(x) = \sum_{k=0}^n s_k x^k$$

on  $\mathbf{G}$  with coefficients  $s_k \in \mathbf{G}$ ,  $s_n \neq 0$ , of degree  $n = \deg S$ . Moreover,  $S(T) = \text{diag}(S(t_1), \dots, S(t_m))$  for the diagonal operator  $T = \text{diag}(t_1, \dots, t_n)$  in the corresponding non-archimedean orthonormal basis in the subspace  $PX_{\mathbf{G}}$  over the field  $\mathbf{G}$ , where  $t_1, \dots, t_m \in \mathbf{G}$ . On the other hand, applying Theorems 5.4, 5.11 and 5.16 [1] we get:

$$(4) \quad \|S(PAP)\| \leq \sup_{t \in \mathbf{G}, |t| \leq \|PAP\|} |S(t)|,$$

since  $\|PAP\| = \sup_{1 \leq v, l \leq m} |q_v^* PAP q_l|$ ,  $\|P\| = 1$  for each non-degenerate projection operator, where  $q_j$  is a non-archimedean orthonormal basis in  $PX_{\mathbf{G}}$ ,  $q_j^* \in PX_{\mathbf{G}}'$  denotes a  $\mathbf{G}$  linear functional corresponding to  $q_j$ .

In view of Kaplansky's theorem a family of polynomials is dense in  $C(B(\mathbf{G}, 0, r), \mathbf{G})$  for each  $0 < r < \infty$  for the locally compact field, since the ball  $B(\mathbf{G}, 0, r)$  is compact. For every  $f \in C_{\infty}(\mathbf{G}, \mathbf{G})$  and each  $r = p^j \in \Gamma_{\mathbf{G}} := \{|x| : x \in \mathbf{G} \setminus \{0\}\}$  a sequence  $\{S_{n_j(k)} : k\}$  of polynomials exists uniformly converging to  $f$  on  $B(\mathbf{G}, 0, p^j)$ , where  $n_j(k) < n_j(k+1)$  for each  $k \in \mathbf{N}$ ,  $n = \deg S_n$ .

By induction construct them such that  $\{n_{j+1}(k) : k \in \mathbf{N}\} \subset \{n_j(k) : k \in \mathbf{N}\}$  for each natural number  $j \in \mathbf{N}$ . Choosing the diagonal subsequence  $\{n_j(j) : j \in \mathbf{N}\}$  one gets a sequence of polynomials  $S_{n_j(j)}$  point wise converging to  $f$  on  $\mathbf{G}$  and uniformly converging to  $f$  on each bounded ball  $B(\mathbf{G}, 0, r)$ , since  $\lim_{|t| \rightarrow \infty} f(t) = 0$ . Since  $\|A\| < \infty$ , the function

$$(5) \quad f(A)x = \lim_{j \rightarrow \infty} \lim_{P \in \mathbf{P}_{\mathbf{G}}} S_{n_j(j)}(C)S_{n_j(j)}(T)S_{n_j(j)}(E)x$$

exists for each  $x \in X$ , where  $C, T$  and  $E$  correspond to  $PAP$ ,  $P \in \mathbf{P}_{\mathbf{G}}$ . Evidently it is linear by  $x \in X$ , since  $\lim_{P \in \mathbf{P}_{\mathbf{G}}} S_{n_j(j)}(C)S_{n_j(j)}(T)S_{n_j(j)}(E)$  is a linear operator on  $X$  over  $\mathbf{F}$  for each  $j$ . Since  $\mathbf{G} \subset \mathbf{F}$ , Formulas (1 – 5) imply that

$$(6) \quad \|f(A)\| \leq \sup_{t \in \mathbf{F}, |t| \leq \|A\|} |f(t)| < \infty.$$

**15. Theorem.** *Suppose that  $\Phi$  is an algebra with transposition of bounded linear operators on a Banach space  $X$  over a spherically complete field  $\mathbf{F}$ , then each  $A \in B(\bar{\Psi}, 0, 1)$  belongs to the strong operator closure  $\overline{B(\Psi, 0, 1)}$  of the unit ball  $B(\Psi, 0, 1)$  of  $\Psi$ . If  $Q$  is a symmetric operator in  $B(\bar{\Psi}, 0, 1)$ , then  $Q$  is in the strong-operator closure of the set of symmetric operators in  $B(\Psi, 0, 1)$ .*

**Proof.** As in section 12 for an absolutely convex subset  $E$  of  $L(X)$  the weak- and strong-operator closures coincide, since  $X$  is a Banach space over a spherically complete field  $\mathbf{F}$ . Indeed, for each proper norm closed linear subspace  $Y$  of  $X$  and a point  $x \in X \setminus Y$  a continuous linear functional  $f : X \rightarrow \mathbf{F}$  exists such that  $f(x) = 1$  and  $f(Y) = 0$  due to the Hahn-Banach theorem over  $\mathbf{F}$  (see §8.203(f) [10]). For each point  $x$  outside the norm closure  $cl_n U$  of a subset  $U$  in  $X$  there exists a closed ball  $B(X, x, r) := \{z \in X : \|z - x\| \leq r\}$  containing  $x$  of radius  $0 < r < \infty$  such that the intersection  $(cl_n U) \cap B(X, x, r) = \emptyset$  is void with  $r \in \Gamma_{\mathbf{F}}$ , where  $\Gamma_{\mathbf{F}} := \{|b| : b \in \mathbf{F} \setminus \{0\}\}$  is a multiplicative group contained in  $\mathbf{R}$ . The multiplicative norm on  $\mathbf{F}$  is non-trivial, consequently, zero is the limit point of  $\Gamma_{\mathbf{F}}$  in  $\mathbf{R}$ . Therefore, a radius  $r > 0$  can be chosen so that  $\inf_{y \in cl_n U} \|x - y\| > r$ .

If  $V$  is an absolutely convex norm closed subset of  $X$  and  $x \in X \setminus V$ , there exists a hyperplane  $y + Y$  in  $X$  which does not contain  $x$  and does not intersect  $V$ , where  $y = \lambda x$  for some  $\lambda \in \mathbf{F}$ ,  $0 < |\lambda| \leq 1$ ,  $X = Y \oplus \mathbf{F}$ .

The topological dual space  $X'$  of all continuous linear functionals  $f : X \rightarrow \mathbf{F}$  separates points in  $X$ , consequently, there exists a family  $\{f_\beta\} \subset X'$  of continuous linear functionals and closed subsets  $K_\beta$  in the field  $\mathbf{F}$  such that  $V = \bigcap_\beta f_\beta^{-1}(K_\beta)$ .

Evidently, if  $A$  is in the strong operator closure of  $E$ , then it is in the weak operator closure of  $E$ . Let now  $A$  be in the weak-operator closure of  $E$ . Consider vectors  $x_1, \dots, x_n \in X$  and the  $n$ -fold direct sum  $X^{\oplus n} = X \oplus \dots \oplus X$ . An operator  $G$  on  $X$  induces an operator  $\tilde{G} = G \oplus \dots \oplus G$  on  $X^{\oplus n}$ . Therefore,  $\{\tilde{G} : G \in E\} =: \tilde{E}$  is an absolutely convex subset of  $X^{\oplus n}$ , hence  $\tilde{E}\tilde{x}$  is an absolutely convex subset of  $X^{\oplus n}$ , where  $\tilde{x} = (x_1, \dots, x_n)$ . If  $\tilde{A}$  is in the weak-operator closure of  $\tilde{E}$ ,  $\tilde{A}\tilde{x}$  is in the weak closure of  $\tilde{E}\tilde{x}$ , hence in the norm closure of  $\tilde{E}\tilde{x}$  in  $X^{\oplus n}$  due to the fact demonstrated above.

That is for each  $\epsilon > 0$  there exist  $T \in E$  such that  $\|Tx_j - Ax_j\| < \epsilon$  for each  $j = 1, \dots, n$ . Thus the weak-operator closure and the strong-operator closure of  $E$  coincide.

In view of Lemma 14, an operator  $f(A)$  is defined for each symmetric bounded operator  $A \in L(X)$  and hence  $f(A)$  for each  $A$  in  $\bar{\Psi}$ , since  $\lim_{|t| \rightarrow \infty} f(t) = 0$ . Moreover, for each bounded symmetric operator  $A$  a symmetric operator  $A_{\mathbf{G}}$  on  $X_{\mathbf{G}}$  corresponds, since  $x^t = x$  for each  $x \in \mathbf{F}$ .

Let  $Q$  be a symmetric operator in  $\bar{\Psi}$ , let also  $K_b$  be a net of operators in  $\Psi$  weak-operator converging to  $Q$ . Then  $(K_b + K_b^t)/2$  is a net of symmetric operators in  $\Psi$  converging to  $Q$  relative to the weak-operator topology. But the set of symmetric operators in  $\Psi$  is absolutely convex and from the fact demonstrated above  $Q$  is in its strong-operator closure.

Consider a symmetric operator  $Q \in B(\bar{\Psi}, 0, 1)$  and a net of symmetric operators  $M_b \in \Psi$  strong-operator converging to  $Q$ . Let  $p$  be a prime number so that  $\mathbf{F}$  is an extension of the  $p$ -adic field  $\mathbf{Q}_p$ , hence up to an equivalence of multiplicative norms on  $\mathbf{F}$  we have  $|p| := |p|_{\mathbf{F}} = 1/p$  (see [14, 15]). Take a continuous function  $f : \mathbf{F} \rightarrow \mathbf{F}$  so that  $f(t) = t$  on  $B(\mathbf{F}, 0, 1)$ , while  $f(t) = p^{2k-1}t$  on  $B(\mathbf{F}, 0, p^k) \setminus B(\mathbf{F}, 0, p^{k-1})$  for each natural number  $k \in \mathbf{N} := \{1, 2, 3, \dots\}$ , since the ball  $B(\mathbf{F}, 0, r)$  is clopen (simultaneously closed and open) in  $\mathbf{F}$ , where  $r > 0$ . The function  $f$  has the natural extension on the field  $\mathbf{K}$  containing  $\mathbf{F}$  so that  $\sqrt[p]{x} \in \mathbf{K}$  for each  $x \in \mathbf{F}$ , putting  $f(t) = t$

on  $B(\mathbf{K}, 0, 1)$ , while  $f(t) = p^{2k-1}t$  on  $B(\mathbf{K}, 0, p^k) \setminus B(\mathbf{K}, 0, p^{k-1})$  for every  $k \in \mathbf{N}$ . Since  $sp(Q) \subset B(\mathbf{K}, 0, 1)$  (see [12, 1]), it follows that  $f(Q) = Q$ . Moreover, the function  $f$  is strong-operator continuous on the set of symmetric operators in  $\bar{\Psi}$ . The inequality  $|f(t)| \leq 1$  for each  $t$  implies that  $\|f(M_b)\| \leq 1$  for each  $b$ . If  $x, y \in \mathbf{K}$  and  $|x - y| < |x|$ , then  $|y| = |x|$  due to the non-archimedean inequality  $|x + y| \leq \max(|x|, |y|)$  for each  $x, y \in \mathbf{K}$ . Therefore,  $f(M_b)$  is strong-operator converging to  $f(Q)$ , since  $\lim_b S_{n_j(j)}(M_b) = S_{n_j(j)}(Q)$  for each  $j$  and  $P \in \mathbf{P}_{\mathbf{G}}$ . Thus  $Q$  is in the strong-operator closure of the set of symmetric operators from  $B(cl_n \Psi, 0, 1)$  and hence the strong operator limit of symmetric elements in  $B(\Psi, 0, 1)$ .

Generally if  $A \in B(\bar{\Psi}, 0, 1)$ , then form an operator  $A' := \begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$  on  $X \oplus X$  which is symmetric. Then  $A' \in B(\bar{\Psi}_2, 0, 1)$ , where  $\Psi_2$  denotes the family of all operators on  $X \oplus X$  presented as  $2 \times 2$  matrices with entries in  $\Psi$ . From the proof above it follows that  $A'$  is in the strong-operator closure of  $\Psi_2$ . Particularly each entry of  $A'$  is in the strong-operator closure of  $B(\Psi, 0, 1)$ , since each entry in  $B(\Psi_2, 0, 1)$  is in  $B(\Psi, 0, 1)$ .

**16. Definition.** A derivation  $D$  of a subalgebra  $\Upsilon$  in  $L(X)$  is called spatial, if an operator  $B \in L(X)$  exists such that  $D = ad B|_{\Upsilon}$ .

**17. Theorem.** Let  $\Psi$  be a  $T$ -algebra on a Banach space over a spherically complete field  $\mathbf{F}$ , let also  $D$  be a derivation of  $\Psi$ . Then for each commutative  $W^t$ -subalgebra  $\Phi$  in a commutant  $\Psi'$  a bounded  $\mathbf{F}$ -linear operator  $B = B_{\Phi} \in L(X)$  exists such that  $B$  commutes with  $\Phi$  and  $D = ad B|_{\Psi}$ .

**Proof.** Evidently  $\Upsilon'$  from Definition 13 is weakly closed in  $L(X)$ , particularly,  $\Psi'$  is weakly closed. Let  $\Xi$  be a maximal commutative subalgebra of  $\Psi'$ , hence it is weakly closed in the commutant  $\Psi'$ . Consider a lattice  $\mathcal{P}$  of projection operators in  $\Xi$  which corresponds to  $\Psi$  (see Theorems 5.4, 5.11 and 5.16 in [1]).

The central carrier of an operator  $A \in \Psi$  is defined to be  $(I - P)$ , where  $P = \bigcup_{\beta} P_{\beta}$  and  $P_{\beta}$  is from the set of all central projections in  $\Psi$  such that  $P_{\beta}A = 0$ , i.e. every  $P_{\beta}$  is in the center  $Z(\Psi)$  of  $\Psi$ . Denote by  $C_A$  the central carrier of  $A$ , then  $C_A A = A$ , since  $A$  is continuous and  $Ax$  is orthogonal to the range of  $P_{\beta}$  for each  $\beta$ , but  $Range(P_{\beta}) \subset Range(P)$ .

Suppose that  $B_{j,k} \in \Psi$  and  $Q_{j,k} \in \Psi'$  are operators, then

(i)  $\sum_k B_{j,k} Q_{k,l} = 0$  if and only if central operators  $A_{j,k} \in \Psi$  exist satisfying the properties:

(ii)  $\sum_k B_{j,k} A_{k,l} = 0$  and  $\sum_k A_{j,k} Q_{k,l} = Q_{j,l}$  for each  $j, l = 1, \dots, n$ . Particularly,  $BQ = 0$  for  $B \in \Psi$  and  $Q \in \Psi'$  if and only if  $C_B C_Q = 0$ .

Indeed, from the properties

$$\sum_k B_{j,k} A_{k,l} = 0 \text{ and } \sum_k A_{j,k} Q_{k,l} = Q_{j,l}$$

of central operators  $A_{j,k} \in \Psi$  it follows that

$$\sum_k B_{j,k} Q_{k,j} = \sum_k B_{j,k} \sum_t A_{k,t} B_{t,j} = \sum_t \sum_k B_{j,k} A_{k,t} B_{t,j} = 0.$$

On the other side, if  $\sum_k B_{j,k} Q_{k,l} = 0$ , then one can consider the ring  $Mat_n(\Psi')$  of all  $n \times n$  matrices with entries in  $\Psi'$  and the union of all projections  $T_n = (A_{j,k})$  in  $Mat_n(\Psi')$  which are annihilated under the left multiplication  $BT_n = 0$  by  $B$ , where  $A_{j,k} \in \Psi'$  for each  $j, k$ . Consider a diagonal matrix  $E_n$  with entries being projections in  $\Psi'$ . Then  $BE_n T_n = 0$ , consequently,  $T_n E_n T_n = E_n T_n$  and hence  $T_n E_n = (T_n E_n T_n)^t = E_n T_n$ . Thus  $A_{j,k} \in Z(\Psi')$ . Then the equality  $BQ = 0$  implies  $T_n Q = Q$ , that is,  $\sum_k A_{j,k} Q_{k,l} = Q_{j,l}$  for each  $j, l = 1, \dots, n$ . Particularly, if  $C_B C_Q = 0$ , then  $BQ = BC_B C_Q Q = 0$ . When  $BQ = 0$ , a central projection  $P$  in  $\Psi$  exists such that  $PB = 0$  and  $PQ = Q$ , consequently,  $PC_B = 0$  and  $Range(C_B) \subset Range(P)$ , hence  $C_B C_Q = 0$ .

Recall that vectors  $y_1, \dots, y_n, \dots$  are called mutually orthogonal in the non-archimedean sense, if  $\|t_1 y_1 + \dots + t_k y_k\| = \max_{j=1}^k \|t_j y_j\|$  for each  $t_1, \dots, t_k \in \mathbf{F}$  and  $k \in \mathbf{N}$ . Two subspaces  $U$  and  $W$  in a normed space  $Y$  are called orthogonal and denoted  $U \perp W$  if each vector  $x \in U$  is orthogonal to every vector  $y \in W$ ,  $x \perp y$ .

A closed  $\mathbf{F}$ -linear subspace  $U$  in a normed space  $Y$  is complemented, if a closed  $\mathbf{F}$ -linear subspace  $V$  in  $Y$  exists so that  $U \cap V = \{0\}$  and  $U + V = Y$ . It is orthocomplemented if it is complemented and in addition orthogonal  $U \perp V$  to its complement  $V$ .

We say, that  $E_1, \dots, E_j$  are (mutually) complemented, if  $E_l E_k = 0$  for each  $1 \leq l \neq k \leq j$ .

A projection operator  $E : Y \rightarrow Y$  is called an orthoprojection if  $E(Y) \perp E^{-1}(0)$ .



By Theorem 3.9 [12] a closed linear subspace  $U$  of a Banach space  $Y$  is complemented if and only if a projection  $P : Y \rightarrow U$  exists. Theorem 3.10 [12] asserts, that a closed linear subspace  $U$  of a Banach space  $Y$  over a non-archimedean field is orthocomplemented if and only if an orthoprojection  $E$  of  $Y$  on  $U$  exists. In view of Theorems 5.13 and 5.16 [12] each closed linear subspace of a Banach space over a spherically complete field is orthocomplemented. On the other hand, each closed linear subspace of a Banach space over a spherically complete field has an orthogonal basis which can be extended to an orthogonal basis of the entire Banach space. Therefore, without loss of generality we consider the family  $\mathcal{P}$  of all orthoprojections  $E : X \rightarrow X$  ( for short of projections).

Then we define a new operator  $D_1$  by the formula:

$$(iii) \quad D_1(A_1E_1 + \dots + A_nE_n) = \bar{D}(A_1)E_1 + \dots + \bar{D}(A_n)E_n,$$

where  $E_1, \dots, E_n \in \mathcal{P}$ ,  $A_1, \dots, A_n \in \bar{\Psi}$ ,  $n \in \mathbf{N}$ ,  $\bar{D}$  is an extension of  $D$  from  $\Psi$  onto  $\bar{\Psi}$  in accordance with Lemma 12. If  $A_1E_1 + \dots + A_nE_n = 0$ , then from the proof above it follows that central operators  $C_{j,k} \in Z(\bar{\Psi})$  exist so that  $\sum_{k=1}^n C_{j,k}E_k = E_j$  and  $\sum_{j=1}^n A_jC_{j,k} = 0$ . In view of Theorem 10  $\sum_{j=1}^n \bar{D}(A_j)C_{j,k} = 0$ , consequently,  $\sum_j \bar{D}(A_j)E_j = 0$  by (i, ii). This means that  $D_1$  is single-valued. Denote by  $\Phi$  an algebra over  $\mathbf{F}$  of all elements of the form  $A_1E_1 + \dots + A_nE_n$  with  $A_j$  and  $E_j$  as above. It is indeed an algebra, since  $A_jE_jA_kE_k = A_jA_kE_jE_k$  for each  $j, k$ .

The definition of  $D_1$  implies that this operator is  $\mathbf{F}$ -linear and bounded on  $\Phi$  due to Formula (iii). Next we verify, that  $D_1$  is a derivation of  $\Phi$ .

If projections  $E_1, \dots, E_j$  are complemented, take  $F_{j+1} = E_{j+1} - E_{j+1}(E_1 + \dots + E_j)$  and so on by induction. From  $E_l(X) \perp E_l^{-1}(0)$  for each  $l = 1, \dots, j+1$  and  $F_{j+1} = (I - E_1 - \dots - E_j)E_{j+1}$  it follows, that  $(I - E_1 - \dots - E_j)(X) \perp (I - E_1 - \dots - E_j)^{-1}(0)$  and  $(I - E_1 - \dots - E_j)(E_{j+1}X) \perp E_{j+1}^{-1}(I - E_1 - \dots - E_j)^{-1}(0)$ , consequently,  $F_{j+1}$  is also the projection. Then  $A_lE_l + A_{j+1}E_{j+1} = A_l(E_l - E_{j+1}E_l) + (A_l + A_{j+1})E_{j+1}E_l + A_{j+1}(E_{j+1} - E_{j+1}E_j)$  for each  $l \leq j$  by induction, consequently, this induces the decomposition  $A_1E_1 + \dots + A_nE_n = B_1F_1 + \dots + B_nF_n$  with complemented projections  $F_1, \dots, F_n \in \mathcal{P}$  and  $B_1, \dots, B_n \in \bar{\Psi}$ .

When  $E_1, \dots, E_n$  are complemented projections and  $x = \sum_{j=1}^n E_jx$  is a

vector in  $X$  of unit norm  $\|x\| = 1$ , then

$$\|(A_1 E_1 + \dots + A_n E_n)x\| = \max_{j=1}^n \|A_j E_j x\|,$$

since  $A_j E_j x = E_j A_j x$  are mutually orthogonal in the non-archimedean sense vectors. Moreover,

$$\|A_j E_j x\| \leq \|A_j E_j\| \|E_j x\| \leq \max_{l=1}^n \|A_l E_l\|,$$

since  $\max_j \|E_j x_j\| = \|x\| = 1$ , hence

$$\|A_1 E_1 + \dots + A_n E_n\| \leq \max_{j=1}^n \|A_j E_j\|. \text{ At the same time}$$

$$\max_j \|A_j E_j\| \leq \|A_1 E_1 + \dots + A_n E_n\|$$

due to the non-archimedean orthogonality of  $E_j$ . This implies

$\|D_1(A_1 E_1 + \dots + A_n E_n)\| = \max_{j=1}^n \|(\bar{D} A_j) E_j\|$ . Considering orthogonal central projections, one gets as a central carrier  $Q_j$  of  $E_j$  in  $\Psi'$  as a projection. Two  $T$ -algebras  $\Psi$  and  $\Theta$  are called  $T$ -isomorphic, if an  $\mathbf{F}$ -linear multiplicative bijective surjective mapping  $\theta : \Psi \rightarrow \Theta$  exists continuous together with its inverse mapping,

$\theta(\alpha B) = \alpha \theta(B)$ ,  $\theta(AB) = \theta(A)\theta(B)$  and  $\theta(A^t) = [\theta(A)]^t$  for each  $\alpha \in \mathbf{F}$ ,  $A, B \in \Psi$ . Since  $\theta$  and  $\theta^{-1}$  are continuous and multiplicative, then  $\|\theta(A)\| = \|A\|$  is an isometry, since

$$\|\theta(\alpha A^n)\| = |\alpha|^n \|\theta(A)\|^n \leq |\alpha|^n \|\theta(A)\|^n$$

for each  $A \in \Psi$  and  $\alpha \in \mathbf{F}$ . The  $T$  algebras  $\bar{\Psi} E_j$  and  $\bar{\Psi} Q_j$  are  $T$ -isomorphic, since  $E^j$ ,  $I - E_j$ ,  $Q_j$ ,  $I - Q_j$  and  $E_j^t$ ,  $(I - E_j)^t = I - E_j^t$ ,  $Q_j$ ,  $I - Q_j^t \in \Psi$ , where  $\Psi$  is topologically complete. Then  $\|(\bar{D} A_j) E_j\| = \|(\bar{D} A_j) Q_j\| = \|\bar{D}(A_j Q_j)\| \leq \|\bar{D}\| \|A_j Q_j\| = \|\bar{D}\| \|A_j E_j\|$ . From Theorem 10 it is known that  $\bar{D}$  annihilates the center  $Z(\bar{\Psi})$  of  $\bar{\Psi}$ . Therefore,

$$\|D_1(A_1 E_1 + \dots + A_n E_n)\| \leq \|\bar{D}\| \max_j \|A_j E_j\| = \|\bar{D}\| \|A_1 E_1 + \dots + A_n E_n\|,$$

consequently,  $D_1$  is bounded. Thus  $D_1$  has a bounded extension being a derivation  $D_1 : \Upsilon \rightarrow \bar{\Upsilon}$ . In view of Lemma 12 it has a continuous extension  $\bar{D}_1$  defined on  $\bar{\Upsilon}$ .

On the other hand,  $\bar{\Upsilon}$  is a  $T$ -algebra containing  $\bar{\Psi}$  and  $\Xi$ , since it contains  $\mathcal{P}$ , the projection lattice of  $\Xi$ , hence  $\Upsilon' \subset \Psi'$  and  $\Upsilon'$  commutes with  $\Xi$ . But  $\Xi$  is a maximal commutative subalgebra in  $\Psi'$ , we get  $\Upsilon' = \Xi$ .

Recall that a vector  $x \in X$  is topologically cyclic relative to the action of  $\Psi$  for a closed linear subspace  $Y$  over  $\mathbf{F}$  if  $\bar{\Psi}x = \{Ax : A \in \bar{\Psi}\}$  is

everywhere dense in  $Y$ . A subspace  $Y$  is called invariant relative to  $\bar{\Psi}$ , if  $AY \subset Y$  for each  $A \in \bar{\Psi}$ . A closed linear subspace  $Y$  in  $X$  over  $\mathbf{F}$  is called topologically irreducible relative to  $\bar{\Psi}$ , if  $Y$  is invariant relative to  $\bar{\Psi}$  and each non zero vector  $x \in Y \setminus \{0\}$  is topologically cyclic relative to  $\bar{\Psi}$ . If  $Y$  is a topologically irreducible subspace, it has an orthocomplement  $X \ominus Y$ . So  $X \ominus Y$  has another topologically invariant subspaces and the process can be done by transfinite induction (see [2]). Therefore, the sum of all topologically irreducible subspaces in  $X$  relative to  $\bar{\Psi}$  is everywhere dense in  $X$ .

For any topologically irreducible subspace  $Y$  relative to  $\bar{\Psi}$  consider the restriction  $\bar{\Psi}|_Y = \{A|_Y : A \in \bar{\Psi}\}$ . Since  $\bar{D}_1 A \in \bar{\Psi}$  for each  $A \in \bar{\Psi}$ , the subspace  $Y$  is invariant relative to  $\bar{D}_1 \bar{\Psi}$  also. The algebra  $\Psi$  and the Banach space  $X$  are over the spherically complete field  $\mathbf{F}$ . Take an (ortho)projection  $P$  from  $X$  onto a finite dimensional over  $\mathbf{F}$  subspace  $PX$  of a topologically irreducible subspace  $Y$ . This induces the finite dimensional over  $\mathbf{F}$  subalgebra  $P\Psi P = \{PAP : A \in \Psi\}$ . Then the differentiations  $PDP : P\Psi P \rightarrow P\Psi P$  and  $PD_1 P : P\Psi P \rightarrow P\Psi P$  act on it.

Let  $J_P$  be the center of  $P\Psi P$ . Then the differentiation operator  $PDP$  annihilates  $J_P$  due to Theorem 10 and hence  $PD_1 P$  annihilates  $J_P \cap \Upsilon$ , consequently,  $PDP$  and  $PD_1 P$  are defined on the quotient algebras  $(P\Psi P)/J_P$  and  $(P\Upsilon P)/J_P$  correspondingly. Introduce on  $(P\Psi P)/J_P$  the Lie algebra  $\Psi_P$  structure by  $[A, B] = AB - BA$  for each  $A, B \in (P\Psi P)/J_P$ . Traditionally  $ad B$  denotes  $ad B(A) = [B, A]$  for each  $A \in L(X)$ . The latter Lie algebra  $\Psi_P$  is non degenerate, i.e. has a non degenerate Killing form  $tr(adA adB)$ , where  $(adA)(E) = [A, E]$  for each  $A, E \in \Psi_P$ . Then  $PDP$  is the differentiation of the Lie algebra  $\Psi_P$  so that  $PDP[A, B] = [PDPA, B] + [A, PDPB]$  and analogously  $PD_1 P$  is the differentiation of  $\Upsilon_P$ . In view of Theorem 1.5.8 [3] the Lie algebra  $\Psi_P$  is complete, i.e. its center is zero and each its differentiation is internal,  $der(\Psi_P) = ad(\Psi_P)$ , also  $\Upsilon_P$  is complete. Thus  $PDP$  and  $PD_1 P$  are internal derivations of  $P\Psi P$  and  $P\Upsilon P$  respectively.

Particularly, if  $\mathbf{F}$  is a locally compact field take  $G_\alpha = \mathbf{F}$ . Generally we consider a family  $\{\mathbf{G}_\alpha : \alpha \in \mu\}$  of locally compact subfields such that  $\overline{\bigcup_{\alpha \in \mu} \mathbf{G}_\alpha} = \mathbf{F}$ . Since  $\mathbf{F}$  is spherically complete and  $\mathbf{G}_\alpha$  is locally compact, then  $\mathbf{G}_\alpha$  is spherically complete. This family of subfields is naturally directed

by inclusion which induces a direction on  $\mu$  such that  $\alpha \leq \beta$  if and only if  $\mathbf{G}_\alpha \subset \mathbf{G}_\beta$ . Consider  $\Psi$  over  $\mathbf{G}_\alpha$  and denote it by  $\Psi_\alpha$ . In view of Alaoglu-Bourbaki's theorem (see §9.202 [10]) each bounded closed ball  $B((\Psi_\alpha)', z, r)$  of radius  $0 < r < \infty$  and containing  $z$  in  $(\Psi_\alpha)'$  is weak-operator closed, since  $\mathbf{G}_\alpha$  is a locally compact field.

From the proof above it follows that  $\text{der}(\Psi_{\alpha,P}) = \text{ad}(\Psi_{\alpha,P})$  for each  $\alpha \in \mu$  and  $P$  as above on  $X_\alpha$ , where  $X_\alpha$  is the Banach space  $X$  considered over  $\mathbf{G}_\alpha$ . The set  $\mathcal{P}_\alpha$  of projections  $P$  on  $X_\alpha$  is also directed by  $P \leq Q$  if and only if  $P(X_\alpha) \subset Q(X_\alpha)$ . There are natural connecting continuous  $\mathbf{G}_\alpha$ -linear mappings  $\pi_\alpha^\beta : X_\beta \rightarrow X_\alpha$  for each  $\alpha \leq \beta \in \mu$ . Put  $B_\alpha$  to be the projective limit  $B_\alpha = \varprojlim_{\mathcal{P}_\alpha} B_{\alpha,P}$  which exists in  $(\Psi_\alpha)'$ . Then we put  $B = \varprojlim_{\mu} B_\alpha$ . These projective limits exist relative to the weak-operator topology due to Proposition 2.5.6 and Corollary 2.5.7 [2]. This operator  $B$  is  $\mathbf{F}$  linear, since it is  $\mathbf{G}_\alpha$  linear on  $X_\alpha$  for each  $\alpha$  and  $\overline{\bigcup_{\alpha \in \mu} \mathbf{G}_\alpha} = \mathbf{F}$ .

Considering all possible topologically invariant subspaces and all (ortho)projections  $P$  with finite dimensional over  $\mathbf{F}$  ranges one gets due to Theorem 15, that  $\bar{D}_1$  is the internal derivation of  $\bar{\Upsilon}$ , since the family of all finite dimensional over  $\mathbf{F}$  subalgebras  $P\Upsilon P$  is everywhere dense in  $\bar{\Upsilon}$  relative to the weak-operator topology. Then  $D = \text{ad}B$  on  $\Psi$  for some  $B \in \Xi'$ , since  $BT - TB = \bar{D}_1(T) = D(I)T = 0$  for each  $T \in \mathcal{P}$ .

**18. Definition.** A derivation  $D$  of an algebra  $\Psi$  is called inner, if  $D = \text{ad}B|_\Psi$  for some element  $B \in \Psi$  of this algebra.

**19. Lemma.** Each derivation  $\text{ad}B$  of a  $T$  algebra  $\Psi$  induces a derivation of  $\Psi'$ . A derivation  $\text{ad}B$  of  $\bar{\Psi}$  is inner if and only if it induces an inner derivation of  $\Psi'$ .

**Proof.** For every  $A \in \Psi$  and  $T \in \Psi'$  one gets  $(BT - TB)A - A(BT - TB) = BTA - TBA - ABT + ATB = (BA - AB)T - T(BA - AB) = 0$ , since  $[B, A] \in \Psi$ . In the case when  $\text{ad}B$  induces an inner derivation of  $\bar{\Psi}$  so that  $\text{ad}B = \text{ad}E$  on  $\bar{\Psi}$  with  $E \in \bar{\Psi}$  this implies that  $(B - E)$  commutes with  $\bar{\Psi}$ . Therefore,  $(B - E) \in \Psi'$ . The inclusion  $E \in \bar{\Psi}$  implies that  $\text{ad}(B - E) = \text{ad}B$  on  $\Psi'$ . That is  $\text{ad}B$  induces an inner derivation of  $\Psi'$ .

**20. Definitions.** Suppose that  $X$  is a Banach space over a field  $\mathbf{F}$  and  $P$  is a projection on  $X$ ,  $P : X \rightarrow X$ , and  $\Psi$  is a  $W^t$  subalgebra

in  $L(X)$ ,  $P \in \Psi$ . A projection  $P$  is called cyclic in  $\Psi$  (or under  $\Psi'$ ), if  $PX = cl_X \text{span}_{\mathbf{F}} \Psi' x$  for some vector  $x \in X$ , where  $\text{span}_{\mathbf{F}} U := \{y \in X : y = b_1 x_1 + \dots + b_n x_n; b_1, \dots, b_n \in \mathbf{F}, x_1, \dots, x_n \in U\}$ ,  $cl_X U$  denotes the closure of a subset  $U$  in  $X$  relative to the norm topology. Such vector  $x$  is called a generating vector under  $\Psi'$ .

An orthoprojection  $P$  in  $\Psi$  over a spherically complete field  $\mathbf{F}$  is called countably decomposable relative to  $\Psi$ , if every orthogonal family of non zero suborthoprojections of  $P$  in  $\Psi$  is countable. When the unit operator  $I$  is countably decomposable relative to  $\Psi$ , one says that the  $W^t$  algebra  $\Psi$  is countably decomposable.

**21. Lemma.** *Let  $P$  be a central (ortho)projection in a  $W^t$  algebra  $\Psi$  over a spherically complete field  $\mathbf{F}$ . This projection  $P$  is the central carrier of a cyclic projection in  $\Psi$  if and only if  $P$  is countably decomposable relative to the center  $Z(\Psi)$  of  $\Psi$ . Moreover, a cyclic projection in  $\Psi$  is countably decomposable; two projections  $P$  and  $Q$  with the same generating vector in  $\Psi$  and  $\Psi'$  have the same central carrier.*

**Proof.** Consider a central projection  $T$  in  $\Psi$  with generating vector  $x \in X$  and  $P = C_T$ . Consider the case when there are orthogonal families  $P_\alpha$  and  $T_\beta$  of (ortho)projections in  $Z(\Psi)$  and  $\Psi$  respectively contained in  $P$  and  $T$  correspondingly. The field  $\mathbf{F}$  is spherically complete and the Banach space  $X$  is isomorphic with  $c_0(\omega, \mathbf{F})$  for some set  $\omega$ . Each closed linear subspace in  $X$  has an orthonormal basis which can be completed to an orthonormal basis in  $X$ . If  $y \in X$ , then there are convergent series  $y = \sum_\alpha P_\alpha y$  and  $y = \sum_\beta T_\beta y$ , where  $P_\alpha y \perp P_\beta y$  and  $T_\alpha y \perp T_\beta y$  are orthogonal in the non-archimedean sense for each  $\alpha \neq \beta$ . The convergence of these series is equivalent to that for each  $\epsilon > 0$  sets  $\{\alpha : \|P_\alpha y\| > \epsilon\}$  and  $\{\beta : \|T_\beta y\| > \epsilon\}$  are finite. Thus these series may have only countable sets of non zero additives.

When  $T_\beta y = 0$  the equalities  $\{0\} = cl_X \text{span}_{\mathbf{F}} \Psi' T_\beta y = cl_X \text{span}_{\mathbf{F}} T_\beta \Psi' y$  and  $T_\beta T y = T_\beta y$  are valid, if  $P_\alpha y = 0$  analogously  $P_\alpha y = 0$ . That is  $P_\alpha P y = P_\alpha y = 0$  due to the equivalence of conditions (i) and (ii) in Section 17. Thus the families  $\{P_\alpha\}$  and  $\{T_\beta\}$  have at most countable subsets of non zero elements, consequently,  $P$  and  $T$  are countably decomposable.

On the other hand, if  $P$  is countably decomposable and  $\{P_n\}$  is a count-

able set of projections cyclic under  $(Z(\Psi))'$  with generating vectors  $x_n$  of unit norm and with sum  $\vee_n P_n = P$ . The field  $\mathbf{F}$  is of zero characteristic and contains the  $p$ -adic field  $\mathbf{Q}_p$  for some prime number  $p$ . Take the vector  $x = \sum_n p^n x_n$ , where  $n \in \mathbf{N}$ . This sum or series converges, since  $\|p^n x_n\| = \|x_n\|p^{-n}$  up to an equivalence of norms on  $\mathbf{F}$ . Therefore the equality is valid  $cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'x = PX$ , since  $cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'x$  contains  $cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'P_n x = cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'x_n = P_n X$  for each  $n$ . Putting  $T$  to be an projection from  $X$  onto  $cl_X \text{ span}_{\mathbf{F}}(\Psi)'x$  one gets  $T \subseteq P$ , since  $\Psi' \subseteq (Z(\Psi))'$ , that is  $PT = T$ . Suppose that  $Q \in Z(\Psi)$  and  $QT = T$ . This implies that  $Qx = x$  and  $cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'x = cl_X \text{ span}_{\mathbf{F}}(Z(\Psi))'Qx = cl_X \text{ span}_{\mathbf{F}}Q(Z(\Psi))'x$ , consequently,  $P = QP$ . This means that  $P = C_T$  with  $T$  cyclic in  $\Psi$ . Therefore, the projection  $P$  is the central carrier of  $cl_X \text{ span}_{\mathbf{F}}\Psi x$ .

**22. Theorem.** *If  $\Psi$  is a  $W^t$  algebra on a Banach space over a spherically complete field  $\mathbf{F}$  and  $D$  is a derivation of  $\Psi$ , then  $D$  is inner.*

**Proof.** In view of Theorem 17 a derivation  $D$  has the form  $D = adB|_{\Psi}$  for some bounded linear operator  $B \in L(X)$ . Then  $-(BA^t - A^t B)^t = B^t A - AB^t \in \Psi$  for each  $A \in \Psi$ , consequently, the mapping  $adB^t : \Psi \rightarrow \Psi$  is also the differentiation of  $\Psi$ . Therefore,  $ad(B+B^t)$  and  $ad(B-B^t)$  are derivations of  $\Psi$ . If each of these derivations  $ad(B+B^t)$  and  $ad(B-B^t)$  is inner, then  $adB$  is inner as well. Mention that the operator  $ad(\lambda I + B)$  is the derivation together with  $adB$  for each  $\lambda \in \mathbf{F}$ . In accordance with Theorem 10  $B\Phi'$  is the center of the  $W^t$  algebra  $\Psi$ , where  $\Phi = Z(\Psi)$ .

If  $\{P_{\beta} : \beta \in \Lambda\}$  is a family of projections on  $X$  so that its sum  $I = \sum_{\beta \in \Lambda} P_{\beta}$  is the unit operator and  $adB|_{\Psi P_{\beta}} = adE_{\beta}|_{\Psi P_{\beta}}$  for every  $\beta$  and  $\sup_{\beta \in \Lambda} \|E_{\beta}\| < \infty$ , where  $\Lambda$  is a suitable set,  $E_{\beta} \in \Psi P_{\beta}$ , then  $adB|_{\Psi} = adE|_{\Psi}$  for  $E = \sum_{\beta \in \Lambda} E_{\beta}$ .

Take  $Q_{\alpha}$  a cyclic projection under  $(Z(\Psi))'$  for each  $\alpha$ . It is sufficient to prove this assertion for countably decomposable center  $Z(\Psi)$  due to Lemma 21. For this one takes a cyclic projection  $T$  in  $\Psi'$  with central carrier  $I$  considering the faithful representation  $\Psi T$  of  $\Psi$  on  $T(X)$ . The commutant is  $T\Psi'T$  and so it is sufficient to consider that  $\Psi'$  is countably decomposable.

Let  $\mathbf{G}$  be a locally compact field contained in  $\mathbf{F}$  and consider the spheri-

cally complete field  $\mathbf{F}$  as the Banach space over  $\mathbf{G}$  isomorphic with  $c_0(\omega, \mathbf{F})$  for some set  $\omega$  (see §21). Then the Banach space  $X$  over  $\mathbf{F}$  has the structure of the Banach space  $X_{\mathbf{G}}$  over  $\mathbf{G}$  as well. To each operator bounded linear operator  $A \in L(X)$  a bounded operator  $A_{\mathbf{G}} \in L(X_{\mathbf{G}})$  corresponds. Due to Alaoglu-Bourbaki's theorem (see §9.202 [10]) a closed bounded ball  $B(X_{\mathbf{G}}, x, r) := \{y \in X_{\mathbf{G}} : \|y - x\| \leq r\}$  in  $X_{\mathbf{G}}$  is weakly compact and a bounded closed ball  $B(L(X_{\mathbf{G}}), A, r) := \{C \in L(X_{\mathbf{G}}) : \|C - A\| \leq r\}$  in  $L(X_{\mathbf{G}})$  is compact relative to the weak operator topology, where  $0 < r < \infty$ . Therefore,  $B(L(X_{\mathbf{G}}), A, r) \cap \Psi_{\mathbf{G}}$  is also compact relative to the weak operator topology, where  $\Psi_{\mathbf{G}}$  is the  $W^t$  algebra  $\Psi$  considered over the field  $\mathbf{G}$ , i.e. by narrowing the field from  $\mathbf{F}$  to  $\mathbf{G}$  so that  $\Psi_{\mathbf{G}} \subset L(X_{\mathbf{G}})$ .

A system of algebras  $\{\Psi_P : P \in \mathcal{P}\}$  and a family of locally compact subfields  $\{\mathbf{G}_{\alpha} : \alpha \in \mu\}$  from §17 gives rise to the projective limit decomposition of each operator  $A \in \Psi$  or  $E \in \Psi'$  and for the differentiation operator  $D$  as well, since  $\Psi = \bar{\Psi}$  by the conditions of this theorem. Finally, from Proposition 2.5.6 and Corollary 2.5.7 [2] the assertion follows.

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